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A Characterization of Circle Graphs

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Une suite à doubles occurrences est une suite de lettres, définie à une permutation circulaire près, où toutes les lettres ont deux occurrences. Deux lettres différentes sont entrelacées si leurs occurrences alternent dans la suite.

Le problème résolu içi est la caractérisation de cette relation d'entrelacement qui s'interprète géométriquement comme la caractérisation du graphe d'intersection des cordes d'un cercle.

La caractérisation proposée est de forme analogue à celle de Whitney pour les graphes planaires.

1. Introduction

The importance of circle graphs, also called chord intersection graphs, arises in various fields of combinatorics from planar graphs to sorting problems and continued fractions [1, 8]. Depending on the properties considered, circle graphs appear under several equivalent definitions such as overlap graphs or permutation graphs.

A first characterization of circle graphs was given by J. C. Fournier [2] in terms of ordered sets. In a previous paper [3], we characterized bipartite circle graphs by properties of vertex neighbourhoods. F. Jaeger studied neighbourhood spaces associated with circle graphs [4] which led him to an original characterization of graphic matroids. It appears to us that the properties of the family incident to a vertex provide an operative tool to produce a structural characterization of circle graphs.

Let us recall that the Whitney characterization of planar graphs relies on the existence of a bijection from the set of the vertex cocycles of a given graph onto the set of the 'cyclic paths' of another graph (the faces of the dual graph). In the same manner, our characterization of circle graphs relies on the existence of a bijection of the vertex cocycles of a given graph onto a set of 'cocyclic paths' of another graph which happens to be planar. But we do not assign, as in the Whitney characterization, that the bijection generates an isomorphism of the corresponding spaces. Although the statements of these two characterizations have some similarity, there is no analogy between their proofs.

Indeed, in this paper we study some properties of cocyclic-path intersection graphs. We prove that a connected graph is a circle graph if and only if it is a cocyclic-path intersection graph.

Our results bring to light some relationships between circle graphs and planarity problems. One interest is to provide a definition of circle graphs more closely related to standard concepts of binary matroids.

2. Definitions and Notation

We shall first recall some basic definitions concerning graphs, cocyclic-path intersection graphs and circle graphs.

Let $G=(V, E)$ be an undirected graph.

The *boundary* ∂e of an edge $e \in E$ is either the two distinct vertices to which e is incident, or the empty set if e is a loop. More generally, *the boundary* ∂F of a subset of edges, $F \subset E$, is the set of vertices which are incident to an odd number of edges of F.

The *cocycle* δx of a vertex $x \in V$ of a simple graph (i.e. without loops of multiple edges) is the set of edges incident to x. More generally, a bipartition of the vertices of a graph,

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$V = V_1 \cup V_2$, defines a cocycle δV_1, $(\delta V_1 = \delta V_2)$, constituted by the set of edges of G joining V_1 and V_2.

A walk $P: x_1 \to x_k$ in G is an alternating sequence of vertices and edges $x_1, e_1, \ldots, x_i, e_i, \ldots, e_{k-1}, x_k$ such that the edge e_i is incident to the vertices x_i and x_{i+1}, for $i = 1, \ldots, k-1$. The vertices x_1, x_k are the *endpoints* of the walk. If $x_1 = x_k$, P is said to be a *closed walk*. A *cycle* is the set of edges having an odd number of occurrences in the sequence of edges of a closed walk.

A *path* is a walk such that each vertex and each edge is different from the other vertices and edges of the walk.

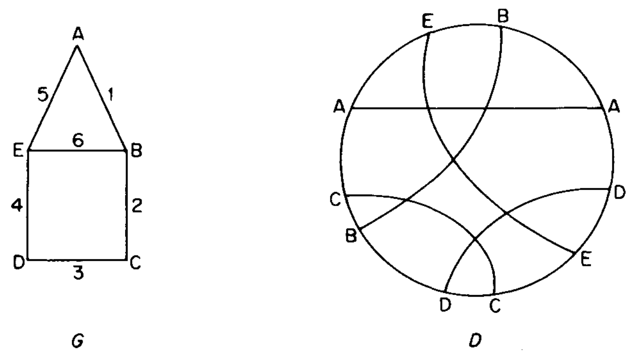
A *cocyclic path* is a path whose set of edges constitutes a cocycle.

There are graphs (see Figure 2) which can be covered by a family of cocyclic paths such that every edge belongs to exactly two cocyclic paths and such that the intersection of any two cocyclic paths is at most an edge. To such graphs, bicovered by cocyclic paths, we associate the corresponding intersection graphs which, by definition, are the cocylic intersection graphs:

A *cocyclic-path intersection graph* is a simple graph with vertex set being a family of cocyclic paths of a given graph, two vertices being adjacent if and only if the corresponding cocyclic paths have an edge in common. Notice that we restrict that definition to graphs covered by cocyclic paths any two of which have at most a common edge.

Let us recall that a *double occurrence sequence* S is a finite sequence of letters on an alphabet E, defined up to a circular permutation, such that each letter has exactly two occurrences in S. Let S^e be the set of letters having exactly one occurrence between both occurrences of e. If e belongs to S^f, then f belongs to S^e. In this case e and f are said to be *interlaced*. The graph of the corresponding binary symmetric relation is the *interlacement graph* of S, denoted by $\Lambda(S)$. By definition, a simple graph is a *circle graph* if it is the interlacement graph of a double-occurrence sequence.

That definition has the following classic geometric interpretation. The letters of the double-occurrence sequence S may be used to label in order, distinct points around the circumference of the unit circle. Each pair of points with the same label are joined by a chord inside the unit circle in such a way that two different chords have at most a common point and three different chords have no common point. To a pair of interlaced letters in S corresponds a pair intersecting chords (see Figure 1). Hence a circle graph can be viewed as a graph with vertex set the set of chords of a family of chords, two vertices being adjacent if and only if the two corresponding chords intersect. By definition, the unit circle and a family of chords, as above, is a *chord diagram*.



S = A, D, E, C, D, B, C, A, E, B

FIGURE 1. A circle graph G and one of its chord diagrams D.

3. Circle Graphs are Cocyclic-Path Intersection Graphs

Let G be a circle graph and D be one of its associated chord diagrams. It is clear that the finite faces of D can be bicoloured. Let $\tilde{G}$ be the adjacency graph of the black faces (see Figure 2).

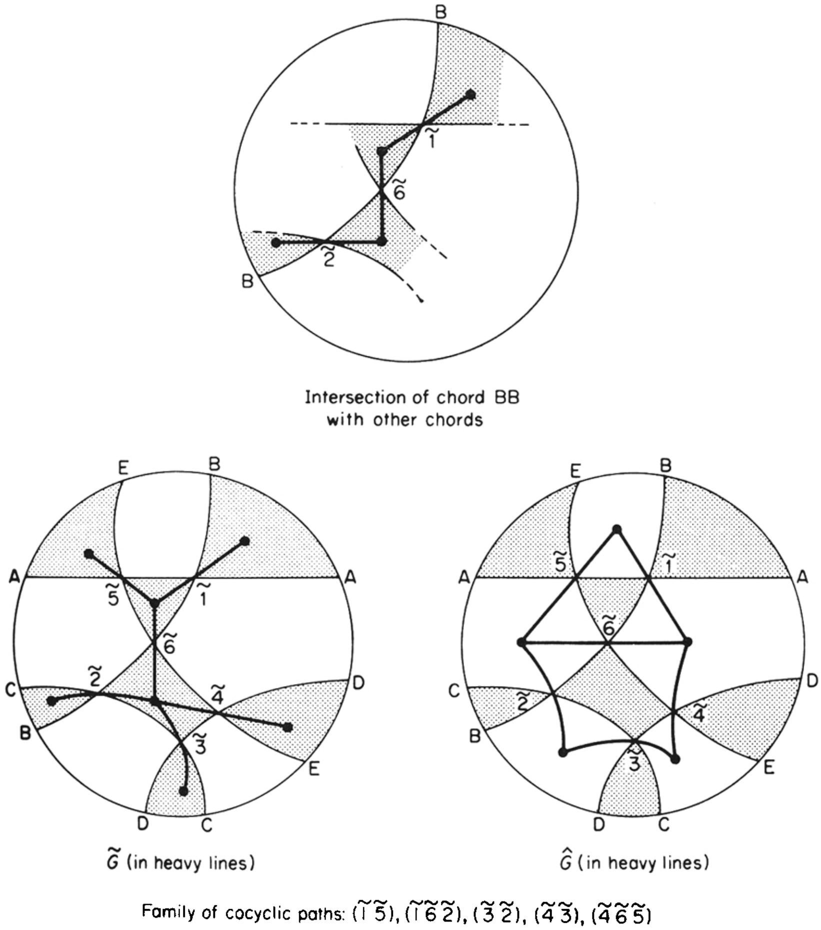


FIGURE 2: $\tilde{G}$ and $\hat{G}$ are the adjacency graphs of respectively the black and the white faces of the chord diagram D associated with the graph G (the same as in Figure 1). Each intersection of the chord BB with another chord, gives rise to an edge in the adjacency graph. As the chord BB partitions the faces of the chord diagram D into two classes: the ones on the left of BB and the others, the set of edges defined by the chord BB is a cocyclic path.

Each intersection of two chords gives rise to an edge in $\tilde{G}$. To the sequence of intersections of a given chord with the others corresponds a path in $\tilde{G}$. Notice that any edge of this path joins a vertex of $\tilde{G}$ (a black face of D) on one side of the chord to a vertex on the other side. Hence, such a path is cocyclic path.

As two chords have at most a common point and intersect if and only if the corresponding vertices in G are adjacent, that construction defines a bijection from the vertex cocycles of G onto a family of cocyclic paths of $\tilde{G}$. Therefore, G is a cocyclic-path intersection graph.

We shall prove that any graph $\tilde{G}$ such that there exists a family of cocyclic paths, with the property that any edge is the intersection of exactly one pair of these cocyclic paths, is a planar graph. Furthermore, we shall prove that the only cocyclic-path intersection graphs are the circle graphs.

4. Results

Theorem 1. *A connected graph is a circle graph if and only if it is a cocyclic-path intersection graph.*

Before starting the proof of this characterization of circle graphs, we shall generalize and restate this theorem in terms of matroid theory. For definitions, we refer the reader to [9].

Theorem 2. *A connected simple graph $G=(V, E)$ is a circle graph if and only if there exists a graphic matroid M on the set $V \cup E$ such that the subsets $(\delta x + x / x \in V)$ are sums of circuits of M and the subsets $(\delta x / x \in V)$ are sums of cocircuits of the matroid $M \backslash V$ deduced from M by the deletion of V.*

Proof. The existence of a graphic matroid M with the properties stated in Theorem 2 is equivalent to the existence of a bijection f from the set of edges of G onto the set of edges of a graph $\tilde{G}$ such that the image of a vertex cocycle $f(\delta x)$ is a cocycle of $\tilde{G}$ whose boundary cardinality is smaller than or equal to two (as by adding an edge, it becomes a cycle). By orthogonality, $f(\delta x)$ is not and does not contain a cycle (otherwise there would exist two vertices $x, y \in V$ such that $f(\delta x) \cap f(\delta y)$ would be a nonempty even set, which contradicts the fact that $\delta x \cap \delta y$ is at most an edge). Hence, the $(f(\delta x)/x \in V)$ defines a family of cocyclic paths and G is a cocyclic-path intersection graph. Therefore the theorems are equivalent.

The same theorem holds under the hypothesis that $M \backslash V$ is a cographic binary matroid.

Given a circle graph G, we have associated with it (Section 3) a planar graph $\tilde{G}$ which allowed us to prove that G was also a cocyclic-path intersection graph. If one looks more closely at $\tilde{G}$, one can remark that the chords of the chord diagram D are pieces of the geometric diagonal of $\tilde{G}$ (see Figure 4).

Conversely, given a cocyclic-path intersection graph, we shall show how the cocyclic paths can be stuck together, and prove that the resulting double-occurrence sequence has the required algebraic properties to provide a constructive solution to the theorem. For that purpose, we shall prove algebraic properties of cocyclic-path intersection graphs, whose geometric interpretation will provide us with the proof.

Let us start with some definitions about diagonals and the principal tripartition of a graph [5].

5. The Principal Tripartition of the Edge Set of a Graph

Let $G=(V, E)$ be an undirected graph and $\mathscr{E}=Z_2^E$, $\mathscr{V}=Z_2^V$ the vector spaces on Z_2 of the subsets of E and V, where the vectorial addition, denoted by $+$, corresponds to the symmetric difference, and where the canonical scalar product is equal to 0 or 1 according to the parity of the intersection of the corresponding subsets. Let ∂ be the boundary operator ∂: $\mathscr{E} \to \mathscr{V}$ associates with any subset A of edges the set of vertices incident to an odd number of edges of A. δ denotes the coboundary operator $\delta: \mathscr{V} \to \mathscr{E}$, i.e. the adjoint operator of ∂ with respect to the canonical scalar product. The cycle space $\mathscr{C}$ of G is by definition the kernel of ∂ and $\mathscr{C}^\perp$ the cocycle space is the image of δ in $\mathscr{E}$ which is exactly the orthogonal space of $\mathscr{C}$ in $\mathscr{E}$. The bicycle space $\mathscr{B}$ is by definition the intersection of $\mathscr{C}$ and $\mathscr{C}^\perp$.

Let R be the set of elements of E contained in at least one bicycle $\beta \in \mathscr{B}$. As $(\mathscr{C} \cap \mathscr{C}^\perp)^\perp = \mathscr{C} + \mathscr{C}^\perp$, every element e not in R can be expressed as the GF(2) sum of a cycle $\gamma(e)$ and a cocycle $\omega(e)$.

By definition $\gamma(e)$ [resp. $\omega(e)$] is a *principal cycle* (resp. *principal cocycle*). The sum of two principal cycles associated with the same edge e is a bicycle. Thus if e belongs to one of its principal cycles it belongs to all its principal cycles. Let P be the set of edges which belong to their principal cycles and Q be the set of edges which belongs to their principal cocycles. The set of edges of a graph is thus partitioned into three classes: P, Q, R.

We shall now recall how those algebraic properties are related to geometrical ones and to our problem.

6. Principal Interlacement Graphs

Let $G=(V, E)$ be a graph with a trivial bicycle space, that is such, that each edge $e \in E$ admits a unique decomposition $e=\gamma(e)+\omega(e)$.

Two edges e, $f \in E$ are *interlaced* if and only if $e \in \gamma(f)$. The graph defined by this binary symmetric relation is called the *principal interlacement graph* and is denoted by $\Lambda(G)$.

P. Rosenstiehl proved that $\Lambda(G)$ is a circle graph if and only if G is planar [5, 3]. That is, there exists a double-occurrence sequence S on E such that $\Lambda(S)=\Lambda(G)$ if and only if G is a planar graph.

S is by definition the *algebraic diagonal* of G. The algebraic diagonal of a planar graph may be defined geometrically as follows:

A *left-right walk* in a planar representation of a graph is a walk such that for any triple (e_1, e_2, e_3) of consecutive edges on the walk (see Figure 3):

(a) $e_1 \neq e_3$,

(b) if e_1, e_2, e_3 are three distinct edges, there exist two faces F_1, F_2 of G such that e_1, e_2 are incident to F_1 and e_2, e_3 are incident to F_2 ($F_1 \neq F_2$ if e_2 is not a bridge). For a complete definition we refer the reader to [5].

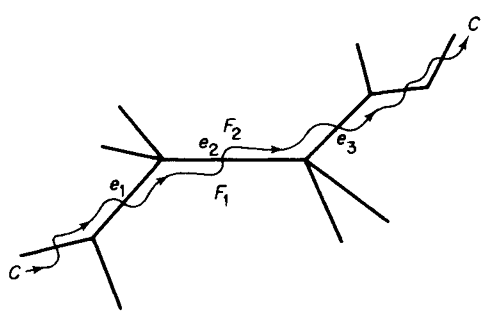
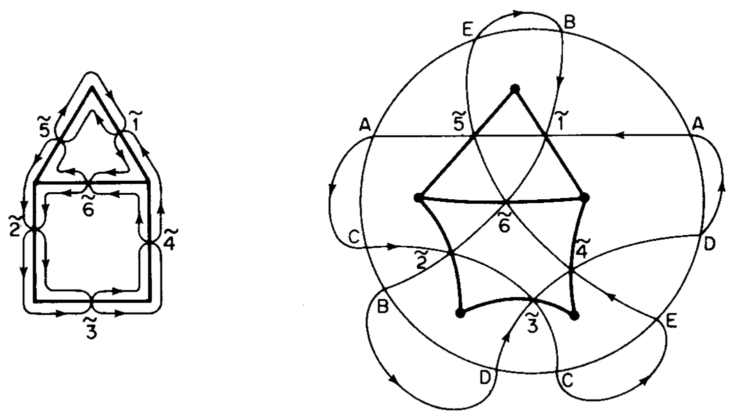


FIGURE 3. A left–right walk C is so called because the sequence of its edges is the sequence of the intersections of a curve C drawn on the plane, whose only crossings are the middle of the edges of G, and such that the curve follows each half of an edge successively in the two faces containing the edge e, that is 'one half of the edge is on the left of C, the other one is on its right'. Here e_1, e_2, e_3 belong to the left–right walk C.

The *geometric diagonal* of a planar representation of a graph (with or without a trivial bicycle space) is, by definition, the set of its left–right walks. If the dimension of the bicycle space of the graph is q, the geometric diagonal consists of $q+1$ closed walks (see Figure 4).

From the work of P. Rosenstiehl on the Gauss problem and on the characterization of planar graphs by their algebraic diagonal, one gets the following theorem:

THEOREM 3 ([5], [6], [7]). *If there exists a walk S in a graph G such that the corresponding double-occurrence sequence is its algebraic diagonal, then G is a planar graph with a trivial*



$$\tilde\Delta = \tilde 1\, \tilde 6\, \tilde 2\, \tilde 3\, \tilde 4\, \tilde 1\, \tilde 5\, \tilde 2\, \tilde 3\, \tilde 4\, \tilde 5\, \tilde 6 = /\tilde 1\, \tilde 6\, \tilde 2/\tilde 3\, \tilde 4/\tilde 1\, \tilde 5/\tilde 2\, \tilde 3/\tilde 4\, \tilde 6\, \tilde 5/ = \tilde C_B\, \tilde C_D\, \tilde C_A\, \tilde C_C\, \tilde C_E$$

FIGURE 4. The cocyclic paths are pieces of the diagonal (heavy lines for $\tilde G$, light lines for the diagonal $\tilde 4$).

bicycle space. Moreover, then there exists a planar representation of G such that S is its geometric diagonal.

Later on, we shall show how it is possible to use this theorem even in the case of a graph with a non-trivial bicycle space. But we can already proceed with the proof of the converse of our theorem, that is that a connected cocyclic-path intersection graph is a circle graph.

7. Some Preliminary Results

Let $G=(V, E)$ be a connected graph and f be a bijection of E onto the set of edges $\tilde E$ of a graph $\tilde G=(\tilde V, \tilde E)$ such that the elementary vertex-cocycles $\{\delta x/x\in V\}$ are in bijection with cocyclic paths $\tilde C_x$ of $\tilde G$: $f(\delta x)=\tilde C_x$.

We shall say that a vertex $\tilde x_t$ of $\tilde G$ is a *terminal vertex* if $\tilde x_t$ is the endpoint of at least one cocyclic path f(δx) for some x of G. The other vertices of G, denoted by $\tilde x_{\text{int}}$, are called *interior vertices.* Similarly a *terminal edge* $\tilde e_t$ is an edge such there exists a cocyclic path $\tilde C_x$ such that $\tilde e_t$ is an edge of $\tilde C_x$ incident to a terminal vertex of $\tilde C_x$.

LEMMA 1. *The image by f^{-1} in G of the cocycle of an interior vertex $\tilde x_{\text{int}}$ of $\tilde G$ is a cycle of G: $f^{-1}(\delta\tilde x_{\text{int}})\in\mathscr{C}(G)$.*

PROOF. An interior vertex $\tilde x_{\text{int}}$ is never, by definition, an endpoint of a cocyclic path $\tilde C_x$. Thus the cocycle $\delta\tilde x_{\text{int}}$ of any interior vertex is orthogonal to every cocyclic path $\tilde C_x$ and so the inverse image $f^{-1}(\delta\tilde x_{\text{int}})$ is orthogonal to every vertex cocycle of G and hence is a cycle.

LEMMA 2. *The inverse image of a cycle of $\tilde G$* is a cycle of G.

PROOF. Any cycle $\tilde\gamma$ of $\tilde G$ is orthogonal to every cocycle of $\tilde G$ and in particular to every cocyclic path. Therefore its inverse image $\gamma=f^{-1}(\tilde\gamma)$ is orthogonal to every vertex cocycle of G and so is a cycle.

LEMMA 3. *The number $\tilde{n}_t$ of terminal vertices of $\tilde{G}$ is at most equal to the number n of vertices of G.*

PROOF. Let H be the bipartite graph $H=(V_1(H), V_2(H), E(H))$ defined as follows:
(a) $V_1(H)$ is the set V of vertices of G;
(b) $V_2(H)$ is the set of terminal vertices of $\tilde{G}$;
(c) $v_1 \in V_1(H)$, $v_2 \in V_2(H)$ are adjacent in H if and only if v_2 is an endpoint of $\tilde{C}_{v_1}$ ($\tilde{v}_2$ is thus a terminal vertex in $\tilde{G}$).

Let $m(H)$ be the number of edges of H. As the degree of a vertex $v_1 \in V_1$ is exactly two, we have:

(1) $m(H) = 2 \text{ Card } V = 2n$.

On the other hand, the sum $\sum_{x \in V} f(\delta x)$ is null, therefore its boundary is null. Thus a terminal vertex $\tilde{x}_t$ is an endpoint of an even number of cocyclic paths. So the degree in H of a vertex $v_2 \in V_2(H)$ is at least equal to two. We have:

(2) $m(H) \geqslant 2 \text{ Card } V_2 = 2\tilde{n}_t$.

From (1) and (2) we have: $n \geqslant \tilde{n}_t$.

The next two lemmas point out two general properties of paths and cocyclic paths. In what follows 'intersection of paths' always means the intersection of the corresponding sets of edges.

LEMMA 4. *Let C_1 and C_2 be two paths of a graph G whose intersection is a path. Then, any cycle γ of G can be expressed as a sum of cycles, each of them having a connected intersection with C_1, C_2 and $C_1 \cup C_2$.*

PROOF. Let C be the union of the paths C_1 and C_2. We shall first prove that γ can be expressed as a sum of polygons (i.e. minimal cycles with respect to inclusion) having a connected intersection with C. As any cycle is a sum of polygons, we may assume that γ is a polygon.

The connected components of $(\gamma \cap C) + \gamma$ (i.e. the set of edges of γ which do not belong to C) are paths q_j, $j \in J$:

$$\gamma = (\gamma \cap C) + \sum_{j \in J} q_j. \tag{1}$$

As C_1 and C_2 have a common edge, C is connected and hence each pair of endpoints of any q_j may be joined by a path r_j contained in C. By construction, for any j, $j \in J$, the cycle $q_j + r_j$ has a connected intersection with C. As q_j and r_j are paths the cycle $q_j + r_j$ is expressed in a unique way as an edge-disjoint union of polygons, and these polygons have a path intersection with C.

The equation (1) can be written:

$$\gamma = \left(\gamma \cap C + \sum_{j \in J} r_j \right) + \sum_{j \in J} (q_j + r_j), \tag{2}$$

where $(\gamma \cup C + \sum_{j \in J} r_j)$ is a cycle contained in C, and hence a sum of polygons contained in C.

Now we shall prove that any polygon β having a path intersection with C can be expressed as a sum of cycles whose intersections with C_1, C_2 and C are paths.

If β is not contained in C and if both endpoints of the path $\beta \cap C$ belong to C_1, we connect those endpoints with a path p_1 of C_1. The cycle β is the sum of the cycle $\beta \cap C + p_1$

contained in C and the cycle $\beta + \beta \cap C_1 + p_1$ whose intersections with C and C_1 are paths and whose intersection with C_2 is also a path as $C_1 \cap C_2$ is a path by hypothesis.

Now we assume that β is either contained in C or that the endpoints of the path $\beta \cap C$ do not belong both to C_1.

Let $\{p_{1,k}\}_{k \in K}$ be the family of maximal paths of $\beta \cap C_1$ having a null intersection with C_2 and whose endpoints are both in C_2.

For any k, the endpoints of a path $p_{1,k}$ are the endpoints of a subpath $p_{2,k}$ of C_2:

$$\beta = \left[\beta + \sum_{k \in K} (p_{1,k} + p_{2,k})\right] + \sum_{k \in K} (p_{1,k} + p_{2,k})$$

If β is contained in C, $\beta + \sum_{k \in K} (p_{1,k} + p_{2,k})$ is a cycle contained in C_2 and hence is null.

If β is not contained in C, the cycle $\beta + \sum_{k \in K} (p_{1,k} + p_{2,k})$ by construction, has a path intersection with C and C_2 and the cycles $p_{1,k} + p_{2,k}$ have also a path intersection with C and C_2. We still have to prove that we can decompose those cycles whose intersection with C_1 is not a path.

Let α be any of those cycle not having a path intersection with C_1. Then, denoting by p the intersection of C_1 and C_2, we have: $\alpha \cap C = p_1 p_2' p p_2''$ where p_1 is a subpath of C_1 and $p_2' p p_2''$ is a subpath of C_2. Let p_1' be a subpath of C_1 connecting the endpoints of p_2'. The cycle α is then decomposed into the cycles $\alpha + p_1'$ and the cycle $p_1' + p_2'$ whose intersections with C_1, C_2 and C are connected, which achieves the proof of the lemma.

By refining the decomposition we could prove that any cycle of G can be expressed as a sum of *polygons* whose intersections with C_1, C_2 and C are paths. Note that if the intersection of C_1 and C_2 was not a path the lemma does not hold.

LEMMA 5. *Let C_1, C_2 be two cocyclic paths of G such that $C_1 \cap C_2$ is exactly one edge e with endpoints x, y. The paths C_1, C_2 can be written: $C_1 = C_2' e C_1''$ and $C_2 = C_2' e C_2''$ where the paths C_1', C_2' are incident to x and C_1'', C_2'' are incident to y. Then $\omega = e + C_1'' + C_2''$ is a cocycle.*

PROOF. We have to prove that any polygon of G is orthogonal to ω. Let γ be any of those polygons. From the previous lemma, we may suppose that the intersection of γ with $C = C_1 \cup C_2$, C_1 and C_2 are paths.

We shall first assume that the edge e does not belong to γ. As $\gamma \cap C_1$ is a path not containing the edge e, we have either $\gamma \cap C_1 = \gamma \cap C_1'$ or $\gamma \cap C_1 = \gamma \cap C_1''$. In the first case $\gamma \cap C_1''$ is empty, in the latter case $\gamma \cap C_1''$ is even as C_1 is a cocycle. In both cases γ is orthogonal to C_1''. Similarly $\gamma \cap C_2''$ is even and so $\gamma \cap \omega = \gamma \cap C_1'' + \gamma \cap C_2''$ is even.

Now we assume that the edge e belongs to γ. As by hypothesis $\gamma \cap C$, $\gamma \cap C_1$ and $\gamma \cap C_2$ are paths, either $\gamma \cap C_1'$ or $\gamma \cap C_2'$ is empty. For example, assume that $\gamma \cap C_2'$ is empty. In that case, as C_2 is a cocycle, $\gamma \cap C_2''$ is an odd set and so $\gamma \cap C_1''$ is empty. We have: $\gamma \cap \omega = \gamma \cap C_2'' + e = \gamma \cap C_2$ and hence is even, which achieves the proof.

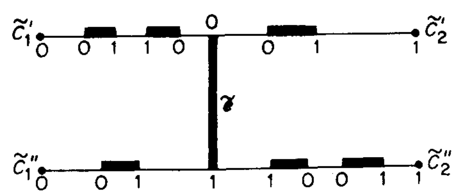


FIGURE 5. In heavy lines, the intersection of a bicycle $\tilde{\beta}$ passing through $\tilde{e}$: both terminal vertices of $\tilde{C}_1$ are at the 0 potential, while both terminal vertices of $\tilde{C}_2$ are at the potential 1.

Lemma 6. *The vector space $\tilde{\mathscr{C}}_{\mathrm{int}}$ generated by the elementary cocycles of interior vertices of $\tilde{G}$ has a null intersection with the cycle space of $\tilde{G}$.*

We have to prove that no non-empty cycle of $\tilde{G}$ is a sum of cocycles of interior vertices of $\tilde{G}$.

Proof. (see Figure 5). Let $\tilde{\beta}$ be a bicycle of $\tilde{G}$ and $\tilde{V}_t$ be the set of terminal vertices of $\tilde{G}$. We shall show that if $\tilde{\beta}=\delta\tilde{V}_0=\delta\tilde{V}_1$ neither $\tilde{V}_t\cap\tilde{V}_0$ nor $\tilde{V}_t\cap\tilde{V}_1$ is empty. As usual, we define a potential: null in $\tilde{V}_0$, equal to one in $\tilde{V}_1$.
we define a potential: null in $\tilde{V}_0$, equals to one in $\tilde{V}_1$.

Let $\tilde{C}_x$ be a cocyclic path of $\tilde{G}$. The intersection of $\tilde{\beta}$ and $\tilde{C}_x$ is even, so both endpoints of $\tilde{C}_x$ are at the same potential. Hence the bicycle $\tilde{\beta}$ defines a bipartition of the cocyclic paths of $\tilde{G}$: those having their endpoints in $\tilde{V}_0$ and those having their endpoints in $\tilde{V}_1$.

Let $\tilde{e}$ be an edge of $\tilde{\beta}$ and $\tilde{C}_1$, $\tilde{C}_2$ the two cocyclic paths containing $\tilde{e}$. As in Lemma 4, let $\tilde{C}_1=\tilde{C}_1'\tilde{e}\tilde{C}_1''$ and $\tilde{C}_2=\tilde{C}_2'\tilde{e}\tilde{C}_2''$ be the decompositions of the two cocyclic paths $\tilde{C}_1$, $\tilde{C}_2$. Since by that lemma $\tilde{e}+\tilde{C}_1''+\tilde{C}_2''$ is a cocycle, $\tilde{\beta}\cap(\tilde{e}+\tilde{C}_1''+\tilde{C}_2'')$ is even, so the intersection of $\tilde{\beta}$ and $\tilde{C}_1''+\tilde{C}_2''$ is odd which implies that the terminal vertex of $\tilde{C}_1$, on $\tilde{C}_1''$, is at a different potential than the one of $\tilde{C}_2$, on $\tilde{C}_2''$. Hence neither $\tilde{V}_t\cap\tilde{V}_0$ nor $\tilde{V}_t\cap\tilde{V}_1$ is empty.

Lemma 7. *The inverse image under f of a bicycle of $\tilde{G}$ is a bicycle of G.*

Proof. Let $\tilde{\beta}$ be a bicycle of $\tilde{G}$. As in the preceding lemma, $\tilde{\beta}$ defines a potential on the vertices of $\tilde{G}$ such that both endpoints of any cocyclic path are at the same potential. Using the same kind of arguments it is easy to check that an edge $\tilde{e}$ belongs to the bicycle $\tilde{\beta}$ if and only if the two cocyclic paths to which $\tilde{e}$ belongs are in different classes defined by $\tilde{\beta}$. Hence $\tilde{\beta}$ is the sum of the cocyclic paths belonging to one of any of the two classes of cocyclic paths.

The inverse image of $\tilde{\beta}$ under f is a cycle of G (Lemma 2) and, from above, is a cocycle as it can be expressed as the sum of the vertex cocycles in bijection with the cocyclic paths of one of any of the two classes defined by $\tilde{\beta}$.

Lemma 8. *The cycle space $\mathscr{C}$ of G is isomorphic to the direct sum of the cycle space $\tilde{\mathscr{C}}$ of $\tilde{G}$ and the subspace $\tilde{\mathscr{C}}_{\mathrm{int}}$ generated by the cocycles of interior vertices of $\tilde{G}$:*

$$\mathscr{C}\overset{f}{\approx}\tilde{\mathscr{C}}\oplus\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp}.$$

Proof. By Lemma 2, $f^{-1}(\tilde{\mathscr{C}})$ is a subspace of $\mathscr{C}$; from Lemma 1, $f^{-1}(\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp})$ is a subspace of $\mathscr{C}$. From Lemma 5, $\tilde{\mathscr{C}}$ and $\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp}$ have a null intersection, hence $\tilde{\mathscr{C}}+\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp}$ is a direct sum, so we only have to prove that:

$$\dim\mathscr{C}=\dim(\tilde{\mathscr{C}}\oplus\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp}).$$

$\tilde{G}$ is necessarily connected as to two adjacent edges in G correspond two edges in $\tilde{G}$ belonging to the same connected component of $\tilde{G}$ which contains the cocyclic path associated with the vertex of G incident to both edges. As in a connected graph the only linearly dependent set of vertex cocycles is the family of all the vertex cocycles, we have: $\dim\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp}=\tilde{n}_{\mathrm{int}}$. On the other hand, $\dim\tilde{\mathscr{C}}=m-\tilde{n}+1$. So we have:

$$\dim(\tilde{\mathscr{C}}\oplus\tilde{\mathscr{C}}_{\mathrm{int}}^{\perp})=m-\tilde{n}+1+\tilde{n}_{\mathrm{int}}=m-\tilde{n}_t+1.$$

As $f^{-1}(\tilde{\mathscr{C}} \oplus \tilde{\mathscr{C}}^{+\text{int}})$ is a subspace of $\mathscr{C}$ we have:

$$m - \tilde{n}_t + 1 \leqslant m - n + 1,$$

that is $\tilde{n}_t \geqslant n$.

By Lemma VII.3, $\tilde{n}$ is at most n. Therefore $n = \tilde{n}_t$ and $f^{-1}(\tilde{\mathscr{C}} \oplus \tilde{\mathscr{C}}^{\perp}_{\text{int}})$ has the same dimension at $\mathscr{C}$ and so is equal to $\mathscr{C}$.

LEMMA 9. *A terminal vertex $\tilde{x}_t$ of $\tilde{G}$ is the endpoint of exactly two cocyclic paths.*

PROOF. If one terminal vertex were the endpoint of more than two cocyclic paths, the inequality (2) of Lemma 3 would be strict:

$$m(H) > 2\tilde{n}_t$$

which contradicts the equality (1) of the same lemma:

$$m(H) = 2n,$$

as in the preceding lemma, we have just proved that $n = \tilde{n}_t$.

We shall now prove that $\tilde{G}$ is a planar graph, by exhibiting a double-occurrence sequence which will be the algebraic diagonal either of $\tilde{G}$ or of an homeomorphic graph $\hat{G}$ of $\tilde{G}$ if $\tilde{G}$ has a non-trivial bicycle space.

PROPOSITION 1. *$\tilde{G}$ is a planar graph.*

PROOF. The proof will be carried out in four steps:

(*a*) *Construction of a family of closed walks.* Let $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n$ the cocyclic paths of $\tilde{G}$ in bijection with the n vertices of G. We shall construct a family $\tilde{\Delta}_j (j = 1, \dots, r+1)$ of closed walks on $\tilde{G}$.

The path $\tilde{C}_1$ defines the beginning of the walk $\tilde{\Delta}_1 : \tilde{x}_1 \to \tilde{x}_2$. By lemma 8, $\tilde{x}_2$ is the endpoint of exactly one other cocyclic path, renumbered $\tilde{C}_2$, which allows us to extend the walk defined by $\tilde{C}_1$.

Let $\tilde{C}_1 \tilde{C}_2$ be the new walk. Unless $\tilde{C}_1 \tilde{C}_2$ is a closed walk, we can still extend it in a unique way, as above.

This process is repeated until the last cocyclic path used leads back to the starting point $\tilde{x}_1$:

$$\tilde{\Delta}_1 = \tilde{C}_1 \tilde{C}_2 \dots \tilde{C}_{k_1}.$$

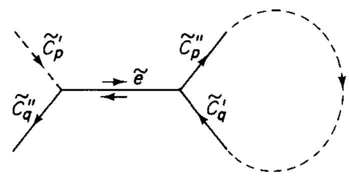
If the cocyclic paths $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_n$ have not all been used for the construction of $\tilde{\Delta}_1$, the previous construction is repeated, starting with any unused cocyclic path.

In this way we obtain a family $\tilde{\Delta}_j (j = 1, \dots, r+1)$ of closed walks such that each cocyclic path $\tilde{C}_i$ is used exactly once.

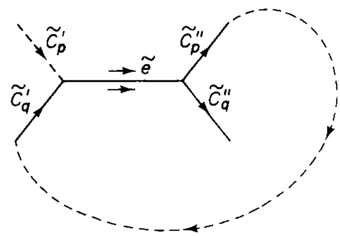
(*b*) *On the interlacement of letters* (see Figure 6). If $\tilde{e}$ has both its occurrences in a walk $\tilde{\Delta}_j$, we shall prove, denoting by $S^{\tilde{e}}$ the edges having exactly one occurrence between two successive occurrences of $\tilde{e}$, that:

$$S^{\tilde{e}} = \tilde{\gamma}(\tilde{e}) \cap \tilde{\omega}(\tilde{e}) \quad \text{where } \tilde{\gamma}(\tilde{e}) \in \tilde{\mathscr{C}}, \tilde{\omega}(\tilde{e}) \in \tilde{\mathscr{C}}^{\perp}.$$

That is, $S^{\tilde{e}}$ is either a cycle and $S^{\tilde{e}} + \tilde{e}$ a cocycle, or $S^{\tilde{e}}$ is a cocycle and $S^{\tilde{e}} + \tilde{e}$ a cycle. $S^{\tilde{e}}$ or $S^{\tilde{e}} + \tilde{e}$ is obviously a cycle and if the edge $\tilde{e}$ is a terminal edge for both cocyclic paths used by $\tilde{\Delta}_j$ it is also obvious that either $S^{\tilde{e}}$ or $S^{\tilde{e}} + \tilde{e}$ is a cocycle. The proof in the



$\tilde{e} \in \tilde{Q}$



$\tilde{e} \in \tilde{P}$

FIGURE 6. For $\tilde{e} \in \tilde{Q}$, if $\tilde{C}'_p = 0$ then $\tilde{C}'_q$ is a cocycle. For $\tilde{e} \in \tilde{P}$, if $\tilde{C}'_p = 0$ then $\tilde{e} + \tilde{C}''_q$ is a cocycle.

general case, is based on Lemma 4. In what follows we suppose that j equals one to simplify the notation. If $\tilde{\Delta}_1$ has letters with just one occurrence in $\tilde{\Delta}_1$, there are two possible sets for $S^{\tilde{e}}$. For example, if $\tilde{\Delta}_1$ is written:

$$\tilde{\Delta}_1 = \tilde{C}_1 \tilde{C}_2 \ldots \tilde{C}_{p-1} \tilde{C}'_p \tilde{e} \tilde{C}''_p, \tilde{C}_{p+1} \ldots \tilde{C}'_q \tilde{e} \tilde{C}''_q, \ldots, \tilde{C}_{k_1},$$

where $\tilde{C}'_p \tilde{e} \tilde{C}''_p = \tilde{C}_p$ and $\tilde{C}'_q \tilde{e} \tilde{C}''_q = \tilde{C}_q$, we arbitrarily choose $S^{\tilde{e}} = \tilde{C}''_p + \tilde{C}_{p+1} + \cdots + \tilde{C}'_q$ (the other possibility would be $S^{\tilde{e}} + \tilde{\Delta}_1$).

Either $\tilde{C}''_p \tilde{C}_{p+1} \ldots \tilde{C}'_q$ or $\tilde{C}''_p \tilde{C}_{p+1} \ldots \tilde{C}_q$, e is a closed walk.

In the first case $S^{\tilde{e}}$ is a cycle:

$$\tilde{\gamma}(\tilde{e}) = \tilde{C}''_p + \tilde{C}_{p+1} + \cdots + \tilde{C}'_q.$$

By Lemma 5, $\tilde{C}''_p + \tilde{C}'_q + \tilde{e}$ is a cocycle and thus $\tilde{\omega}(\tilde{e}) = (\tilde{C}''_p + \tilde{C}'_q + \tilde{e}) + \tilde{C}_{p+1} + \tilde{C}_{p+2} + \cdots + \tilde{C}_{q-1}$ is a cocycle such that:

$$\tilde{e} = \tilde{\gamma}(\tilde{e}) + \tilde{\omega}(\tilde{e}) \qquad \tilde{e} \in \tilde{\omega}(\tilde{e}),\ \tilde{\omega}(\tilde{e}) \in \tilde{\mathscr{C}}^{\perp},\ \tilde{\gamma}(\tilde{e}) \in \tilde{\mathscr{C}},\ \tilde{e} \in \tilde{Q}.$$

Similarly if $\tilde{C}''_p \tilde{C}_{p+1} \ldots \tilde{C}'_q \tilde{e}$ is a closed walk, $\tilde{\gamma}(\tilde{e}) = \tilde{C}''_p + \tilde{C}_{p+1} + \cdots + \tilde{C}'_q + \tilde{e}$ is a cycle and $\tilde{\omega}(\tilde{e}) = \tilde{\gamma}(e) + \tilde{e}$ is a cocycle ($\tilde{e} \in \tilde{P}$).

(*c*) *Proof of the proposition in the case G has a trivial bicycle space.* Let G' be the intersection graph of the closed walks $\tilde{\Delta}_j$, two vertices of G' being adjacent if and only if the corresponding closed walks have a non-empty intersection.

G' is a connected graph as it can be obtained from the graph G by identifying vertices of G corresponding to cocyclic paths used for a same closed walk $\tilde{\Delta}_j$.

Hence if a closed walk $\tilde{\Delta}_1$ is a double occurrence sequence then the family of closed walks $\tilde{\Delta}_j (j = 1, \ldots, r+1)$ is reduced to $\tilde{\Delta}_1$.

The set $\tilde{\beta}_1$ of edges used only once by $\tilde{\Delta}_1$ constitutes a bicycle of $\tilde{G}$, therefore if $\tilde{G}$ has a trivial bicycle space, $\tilde{\beta}_1$ is null and the family of the $\tilde{\Delta}_j$ reduces to $\tilde{\Delta}_1$:

$$\tilde{\Delta}_1 = \tilde{C}_1 \tilde{C}_2 \ldots \tilde{C}_n, \text{ which, from } (b), \text{ is the algebraic diagonal of } \tilde{G}.$$

From P. Rosenstiehl's theorem 3, we deduce that $\tilde{G}$ is a planar graph and $\tilde{\Delta}_1$ is its geometrical diagonal for some planar representation of $\tilde{G}$.

(*d*) *Proof of the proposition in the general case.* If $\tilde{G}$ has a non-trivial bicycle space, one cannot define uniquely its algebraic diagonal. But if the graph is planar and given a planara representation, one can always construct its geometric diagonal which is constituted by $q+1$ closed walks, where q is the dimension of the bicycle space.

The bisection of q edges belonging to q independent bicycles kills the bicycle space. The geometric diagonal of the resulting graph is easily deductible from the $q+1$ closed walks of the geometric diagonal of the original graph.

So we shall first construct from the $r+1$ closed walks $\tilde{\Delta}_1, \dots, \tilde{\Delta}_{r+1}$ one closed walk. This is done in r steps by *chaining* each time two closed walks together.

From above, in each closed walk there exist edges having only one occurrence. Let $\tilde{e}_1$ be an edge used in two different walks $\tilde{\Delta}_1$, $\tilde{\Delta}_2$. Let

$$\tilde{\Delta}_1 = \tilde{C}'_1\tilde{e}_1\tilde{C}''_1\tilde{C}_2 \dots \tilde{C}_{k_1}, \qquad \tilde{\Delta}_2 = \tilde{C}'_{k+1}\tilde{e}_1\tilde{C}''_{k_1+1} \dots \tilde{C}_{k_2},$$

where $\tilde{C}_1 = \tilde{C}'_1\tilde{e}_1\tilde{C}''_1$ and $\tilde{C}_{k_1+1} = \tilde{C}'_{k_1+1}\tilde{e}_1\tilde{C}''_{k_1+1}$.

Let us bisect the edge $\tilde{e}_1$, that is replaced it by a cocyclic path $\tilde{e}'_1\tilde{e}''_1$ having the same endpoints as $\tilde{e}_1$.

We create a new closed walk $\tilde{\Delta}_{1,2}$ by chaining $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ the following way:

$$\tilde{\Delta}_{1,2} = \tilde{e}'_1\tilde{e}''_1\tilde{C}_1 \dots \tilde{C}_{k_1}\tilde{C}'_1\tilde{e}'_1\tilde{e}''_1\tilde{C}''_{k_1+1} \dots \tilde{C}_{k_2}\tilde{C}'_{k_1+1}.$$

It is easy to check that for a letter $\tilde{f}$ which has both its occurrences in $\tilde{\Delta}_{1,2}$ one always has:

$$S^{\tilde{f}} = \tilde{\gamma}(\tilde{f}) \cap \tilde{\omega}(\tilde{f}),$$

and the same result still holds for $\tilde{f} = \tilde{e}'_1$ (resp. $\tilde{e}''_1$) as $\tilde{e}''_1 + \tilde{C}''_1 + \tilde{C}'_1$ is a cocycle in the new graph.

Let $\tilde{g}$ be an edge which was used once in $\tilde{\Delta}_1$ and once in $\tilde{\Delta}_2$. $\tilde{\Delta}_{1,2}$ can be written:

$$\tilde{\Delta}_{1,2} = \tilde{e}'_1\tilde{e}''_1\tilde{C}''_1 \dots \tilde{C}'_p\tilde{g}\tilde{C}''_p \dots \tilde{C}_{k_1}\tilde{C}'_1\tilde{e}'_1\tilde{e}''_1\tilde{C}''_{k_1+1} \dots \tilde{C}'_q\tilde{g}\tilde{C}''_q \dots \tilde{C}_{k_2}\tilde{C}'_{k_1+1}.$$

We still have:

$$S^{\tilde{g}} = \tilde{\gamma}(\tilde{g}) \cap \tilde{\omega}(\tilde{g}), \qquad \tilde{\gamma}(\tilde{g}) \in \mathscr{C}, \qquad \omega(\tilde{g}) \in \mathscr{C}^{\perp}$$

(see Figure 7) as it is easy to prove, using the same arguments as in Lemma 4, that $\tilde{C}'_1 + \tilde{e}'_1 + \tilde{e}''_1 + \tilde{C}''_{k_1+1}$ is a cocyclic path and so is $\tilde{C}''_p + \tilde{C}'_q$ or $\tilde{C}''_p + \tilde{C}'_q + \tilde{g}$ (Lemma 5), depending on whether the edge $\tilde{g}$ is followed twice with the same direction or not.

By proceeding to chain all the walks $\tilde{\Delta}_j$, one obtains a closed walk $\hat{\Delta}$ on a graph $\hat{G}$

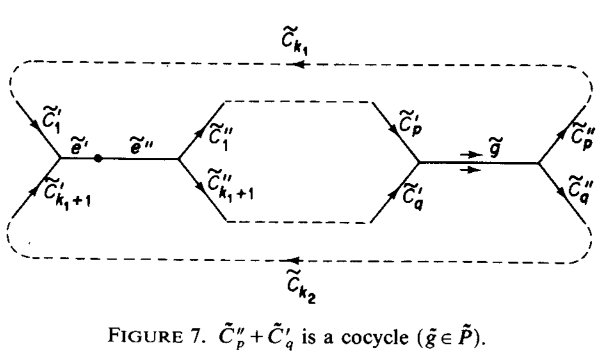


FIGURE 7. $\tilde{C}''_p + \tilde{C}'_q$ is a cocycle ($\tilde{g} \in \tilde{P}$).

deduced from $\tilde{G}$ by r bisections of edges. $\hat{G}$ is therefore a planar graph and $\hat{\Delta}$ is its geometric diagonal for some planar representation. $\tilde{G}$ is hence a planar graph and the family of the $\tilde{\Delta}_j (j = 1, r+1)$ is the geometric diagonal of $\tilde{G}$ for some planar representation of $\tilde{G}$.

8. Proof of the Theorem: Construction of the Chord Diagram

As stated previously, the chords of the chord diagram are pieces of the diagonal of a graph. The candidate graph is $\tilde{G}$ with its planar representation $\tilde{G}_0$ such that the $\tilde{\Delta}_j (j = 1, r+1)$ constitute its geometric diagonal. The $\tilde{C}_i$ $(i = 1, n)$ will be the chords. But we still need a circle. We shall prove that all the terminal edges belong to the same face which we can take to be the exterior face of $\tilde{G}_0$. Whether G has bridges or not, given the embedding $\tilde{G}_0$, one can construct a closed curve surrounding the exterior face such that the edges followed twice are bridges and such that the set of edges followed once is the exterior face (see Figure 8). To the different possible choices of the exterior face correspond such closed curves which defines closed walks $\tilde{\phi}_k$.

We shall first show how to compute the closed walks $\tilde{\phi}_k$ (defined by a curve surrounding a face) and prove that there exists a closed walk $\tilde{\phi}_t$ such that all the terminal edges belong to $\tilde{\phi}_t$ which will define the exterior face of $\tilde{G}_0$.

For clarity, we shall assume that the bicycle space of $\tilde{G}$ is trivial. The general case would be treated by considering the graph $\hat{G}$ with a trivial bicycle space deduced from $\tilde{G}$ by r bisections.

Two consecutive edges in $\tilde{\phi}_t$ are consecutive occurrences in $\tilde{\Delta}$. That property is not sufficient to construct a $\tilde{\phi}_k$ as the occurrences of an edge in $\tilde{\Delta}$ may define four different adjacent occurrences in $\tilde{\Delta}$. But we can get more information.

As has already been mentioned in Section 5, an edge $\tilde{e}$ belongs to the class $\tilde{P}$ or $\tilde{Q}$ of the graph $\tilde{G}$ depending on whether or not it belongs to its principal cycle. So, an edge $\tilde{e}$ belongs to the class $\tilde{P}$ or $\tilde{Q}$ depending on whether or not the walk, defined by the diagonal, follows the edge both times with the same direction or not (cf. case (b) in the proof of Proposition 1).

Hence, if $\tilde{e}$ and $\tilde{f}$ are two edges used consecutively by ϕ_k (and also by $\tilde{\Delta}$), the next consecutive edge used by $\tilde{\phi}_k$ is the edge *following* the other occurrence of $\tilde{f}$ in $\tilde{\Delta}$ if $\tilde{f}$ belongs to $\tilde{P}$, or the edge *preceding* the other occurrence of $\tilde{f}$ in $\tilde{\Delta}$ if $\tilde{f}$ belongs to $\tilde{Q}$.

As an example, for $\tilde{\Delta} = \tilde{e}\tilde{f} \dots \tilde{g}'\tilde{f}\tilde{g}''$, if $\tilde{f} \in \tilde{P}$ then $\tilde{e}, \tilde{f}, \tilde{g}''$ are consecutive edges of $\tilde{\phi}_t$, and if $\tilde{f} \in \tilde{Q}$ then $\tilde{e}, \tilde{f}, \tilde{g}'$ are consecutive edges of $\tilde{\phi}_t$.

We shall not further describe or justify the algorithm for the $\tilde{\phi}$, but refer the reader to [5] for a more formal exposition based on a quadrialphabet associated with the edges of a planar embedding of a graph.

Let $\tilde{\phi}_t$ be the closed walk defined by the previous algorith, starting with two terminal edges $\tilde{e}_0, \tilde{f}_0$ consecutive in $\tilde{\Delta}$. We shall prove that all the terminal edges belong to $\tilde{\phi}_t$.

There is no reason why $\tilde{\phi}$ should consist of terminal edges only and in fact it may not. That is why we shall subdivide the cocyclic paths $\tilde{C}_i$ $(i = 1, n)$ in order to obtain a larger collection of cocyclic paths $\tilde{C}'_j (j = 1, n'; n' \geq n)$ such that $\tilde{\phi}_t$ will consist entirely of terminal edges of the new family of cocyclic paths.

Let $\tilde{e}$ be a terminal edge of a cocyclic path $\tilde{C}_p$. The edge $\tilde{e}$ belongs to another cocyclic path $\tilde{C}_q$. Let $\tilde{C}_p = \tilde{e}\tilde{C}''_p$, $\tilde{C}_q = \tilde{C}'_q\tilde{e}\tilde{C}''_q$ be the two cocyclic paths written in the order they appear for an arbitrary orientation of the diagonal.

By Lemma 4, either $\tilde{C}'_p\tilde{e}$ or $\tilde{C}'_p$ is a cocyclic path depending on whether or not $\tilde{e}$ belongs to $\tilde{P}$. In both cases, if $\tilde{C}''_q$ is not empty, the cocyclic path $\tilde{C}_q$ is subdivided into two cocyclic paths: if $\tilde{e} \in \tilde{P}$ then $\tilde{C}_q \to \tilde{C}'_q\tilde{e}$ and $\tilde{C}''_q$ are cocyclic paths, if $\tilde{e} \in \tilde{Q}$ then $\tilde{C}_q \to \tilde{C}'_q$ and $\tilde{e}\tilde{C}''_q$ are cocyclic paths (see Figure 6).

After the subdivisions of all the cocyclic paths, whenever possible, it appears that a terminal edge of a cocyclic path is also a terminal edge for the other cocyclic path to which it belongs.

For two occurrences of an edge $\tilde{e}$, the diagonal $\tilde{\Delta}$ can be written:

$$\tilde{\Delta} = \ldots ; \tilde{e}\tilde{C}''_p \ldots ; \tilde{C}'_q \tilde{e} ; \tilde{C}''_q ; \ldots , \qquad \text{if } \tilde{e} \in \tilde{P},$$
$$\tilde{\Delta} = \ldots ; \tilde{e}\tilde{C}''_p ; \ldots ; \tilde{C}'_q ; \tilde{e}\tilde{C}''_q ; \ldots , \qquad \text{if } \tilde{e} \in \tilde{Q},$$

where the semicolons delimit the cocyclic paths of the new family.

So, for a terminal edge $\tilde{f}$ we have either:

$$\tilde{\Delta} = \tilde{e} ; \tilde{f} \ldots \tilde{f} ; \tilde{g} \ldots , \qquad \text{if } \tilde{f} \in \tilde{P},$$

that is $\tilde{e}, \tilde{f}, \tilde{g}''$ belongs to $\tilde{\phi}_t$; or

$$\tilde{\Delta} = \tilde{e} ; \tilde{f} \ldots \tilde{g} ; \tilde{f} \ldots , \qquad \text{if } \tilde{f} \in \tilde{Q},$$

that is $\tilde{e}, \tilde{f}, \tilde{g}'$ belong to $\tilde{\phi}_t$.

Thus $\tilde{\phi}_t$ only contains occurrences of the terminal edges defined by the $\tilde{C}'_i$ $(i = 1, n)$. Furthermore if a terminal edge of $\tilde{C}'_i$ belongs to $\tilde{\phi}_t$, by orthogonality, the other terminal edge of $\tilde{C}'_i$ belongs to $\tilde{\phi}_t$ and so it is easy to check that all the terminal edges of the $\tilde{C}'_i$ $(i = 1, n')$ and hence of the $\tilde{C}_i$ $(i = 1, n)$ belong to $\tilde{\phi}_t$.

To effectively construct the chord diagram associated with G, one should first compute the algebraic diagonal $\tilde{\Delta}$ of $\tilde{G}$ (or $\hat{G}$ if $\tilde{G}$ has a non-trivial bicycle space). With Dehn's algorithm [5], one may find a planar representation of $\tilde{G}$ such that $\tilde{\Delta}$ is its geometric diagonal. The chords are then the pieces of the diagonal corresponding to the cocyclic paths associated with the vertices of G. From what we have just proved, one can draw a circle passing through all the endpoints of the diagonal pieces.

Actually, it is easier to deduce directly from $\tilde{\phi}_t$ the double-occurrence sequence S, whose interlacement graph is G:

Let $\tilde{S}$ be the sequence of terminal vertices of $\tilde{G}$ in the order they are encountered by the walk defined by $\tilde{\phi}_t$.

To get to and leave from a terminal vertex $\tilde{x}_{t_i}$, $\tilde{\phi}_t$ uses some edges of two different cocyclic paths. Let $\tilde{C}_{x'_i}$ be the cocyclic path leading to $\tilde{x}_{t_i}$ and $\tilde{C}_{x''_i}$ the cocyclic path leaving from x_{t_i}.

If $\tilde{S} = \tilde{x}_{t_1}\tilde{x}_{t_2}, \ldots \tilde{x}_{t_n}$, the required sequence S such that $\Lambda(S) = G$, can be written:

$$S = x'_1, x''_1, x'_2, x''_2, \ldots , x'_i, x''_i, \ldots , x'_n, x''_n.$$

We conclude by giving an example and a final remark.

9. An Example (See Figures 8 and 9)

$\tilde{G}$ is covered by the following cocyclic paths:

$$\tilde{C}_{\mathrm{A}} = \tilde{1}\,\tilde{7}\,\tilde{6}, \qquad \tilde{C}_{\mathrm{B}} = \tilde{1}\,\tilde{2}\,\widetilde{13}\,\widetilde{12}, \qquad \tilde{C}_{\mathrm{C}} = \tilde{3}\,\tilde{2}\,\tilde{7}\,\tilde{8}, \qquad \tilde{C}_{\mathrm{D}} = \tilde{3}\,\widetilde{13}\,\tilde{4}\,\tilde{9},$$
$$\tilde{C}_{\mathrm{E}} = \tilde{5}\,\tilde{4}\,\widetilde{11}, \qquad \tilde{C}_{\mathrm{F}} = \tilde{8}\,\tilde{6}\,\tilde{5}\,\widetilde{10}, \qquad \tilde{C}_{\mathrm{G}} = \widetilde{10}\,\widetilde{11}\,\widetilde{12}.$$

One can check that the cycle space of G is generated by the faces of $\tilde{G}$:

$$\tilde{1}\,\tilde{2}\,\tilde{7}, \qquad \tilde{2}\,\tilde{3}\,\widetilde{13}, \qquad \tilde{4}\,\widetilde{11}\,\widetilde{12}\,\widetilde{13}, \qquad \tilde{6}\,\tilde{7}\,\tilde{8}, \qquad \tilde{4}\,\tilde{5}\,\tilde{9},$$

and the cocycle of interior vertices of $\tilde{G}$:

$$\tilde{4}\,\tilde{5}\,\tilde{6}\,\tilde{7}\,\tilde{2}\,\widetilde{13}, \qquad \tilde{4}\,\tilde{9}\,\widetilde{10}\,\widetilde{11}.$$

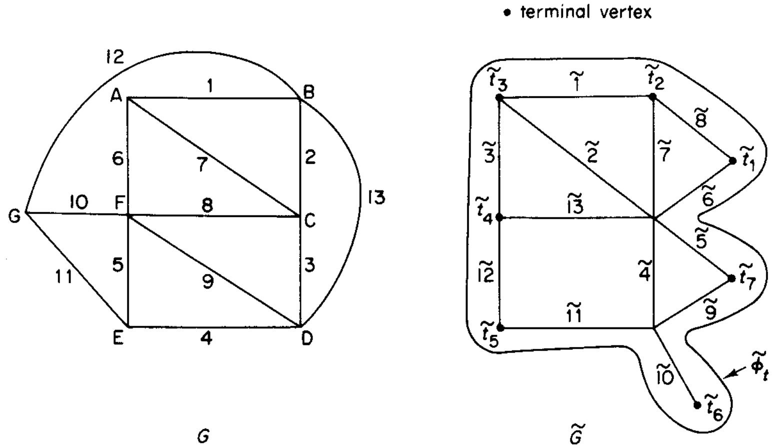


FIGURE 8

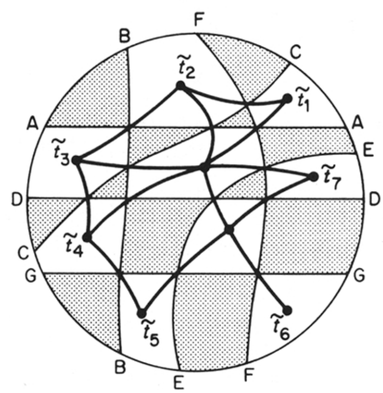


FIGURE 9

The diagonal of $\tilde{G}$, computed by sticking together the cocyclic paths, is the following:

$$\tilde{\Delta} = \tilde{C}_A \tilde{C}_C \tilde{C}_G \tilde{C}_F \tilde{C}_E \tilde{C}_D$$

where the cocyclic paths have been well-oriented. The subdivided sequence $\tilde{\Delta}_S$ is the following:

$$\tilde{\Delta}_S = \tilde{1}\ \tilde{7}\ \tilde{6}/\tilde{8}\ \tilde{7}\ \tilde{2}\ \tilde{3}/\tilde{12}\ \tilde{11}/\tilde{10}/\tilde{10}/\tilde{9}\ \tilde{5}/\tilde{6}\ \tilde{8}/\tilde{1}\ \tilde{2}\ \tilde{13}\ \tilde{12}/\tilde{11}\ \tilde{4}\ \tilde{5}/\tilde{9}\ \tilde{4}\ \tilde{13}\ \tilde{3}.$$

As

$$\tilde{P} = \tilde{4}\ \tilde{6}\ \tilde{7}\ \tilde{8}\ \tilde{11}\ \tilde{12}\ \tilde{13} \quad \text{and} \quad \tilde{Q} = \tilde{1}\ \tilde{2}\ \tilde{3}\ \tilde{5}\ \tilde{9}\ \tilde{10},$$

the walk $\tilde{\Phi}_t$ passing through the terminal edges of $\tilde{G}$ is deduced from $\tilde{\Delta}_S$, the following way:

$$\tilde{\Delta}_S = \underset{\leftarrow}{\underline{\tilde{1}}}\ \tilde{7}\ \underset{\rightarrow}{\underline{\tilde{6}/}}\tilde{8}\ \tilde{7}\ \tilde{2}\ \underset{\rightarrow}{\underline{\tilde{3}/}}\tilde{12}\ \underset{\rightarrow}{\underline{\tilde{11}/}}\underset{\leftrightarrows}{\underline{\tilde{10}/}}\underset{\rightarrow}{\underline{\tilde{10}/}}\tilde{9}\ \underset{\rightarrow}{\underline{\tilde{5}/}}\tilde{6}\ \underset{\rightarrow}{\underline{\tilde{8}/}}\tilde{1}\ \tilde{2}\ \tilde{13}\ \underset{\rightarrow}{\underline{\tilde{12}/}}\tilde{11}\ \tilde{4}\ \underset{\leftarrow}{\underline{\tilde{5}/}}\tilde{9}\ \tilde{4}\ \tilde{13}\ \underset{\leftarrow}{\underline{\tilde{3}/}}$$
$$\tilde{\Phi}_t = \tilde{6}\ \tilde{8}\ \tilde{1}\ \tilde{3}\ \tilde{12}\ \tilde{11}\ \tilde{10}\ \tilde{10}\ \tilde{9}\ \tilde{5}.$$

If we denote by $\tilde{t}_1, \tilde{t}_2, \tilde{t}_3, \ldots, \tilde{t}_7$ the corresponding sequence of terminal vertices, we have, considering the order in which the cocyclic paths were used to build $\tilde{\phi}_t$

$$\begin{array}{lll} x_1' = A, & x_1'' = C, & \\ x_2' = F, & x_2'' = B, & \\ x_3' = A, & x_3'' = D, & + \\ x_4' = C, & x_4'' = G, & \\ x_5' = B, & x_5'' = E, & \\ x_6' = F, & x_6'' = G, & + \\ x_7' = D, & x_7'' = E, & + \end{array}$$

(we have put a + when the orbit $\tilde{\phi}_t$ uses a different orientation to that of the diagonal. For example, $\tilde{C}_{\mathrm{D}}$ leaves $\tilde{t}_7$ and leads to $\tilde{t}_3$, but when used in the diagonal $\tilde{C}_{\mathrm{D}}$ leaves $\tilde{t}_3$ and leads to $\tilde{t}_7$).

Hence the double-occurrence sequence S such that $G = A(S)$, is the following:

$$S = \mathrm{A, C, F, B, A, D, C, G, B, E, F, G, D, E.}$$

10. Remark

In the example given in Figure 2, the adjacency graph of the black faces is a tree and the adjacency graph of the white faces, which also gives a solution for the theorem, has its cycle space isomorphic to the one of G.

That property is not generally true, as one can see by studying the possible chords diagrams associated with the circle graphs $K_{2,3}$ or K_4.

References

1. S. Even and A. Itai, Queues, stacks and graphs, in *Theory of Machines and Computations*, Academic Press, New York, 1971, pp. 71–86.
2. J. C. Fournier, Une caractérisation des graphes de cordes, *C.R. Acad. Sci. Paris* **286A** (1978), 811–813.
3. H. de Fraysseix, Local complementation and interlacement graphs, *Discrete Math.* **33**(1) (1981), 29–35.
4. F. Jaeger, Graphes de cordes et espaces graphiques, (to appear).
5. P. Rosenstiehl, Caractérisation des graphes planaires par une diagonale algèbrique, *C.R. Acad. Sci. Paris* **283A** (1976), 417–419.
6. P. Rosenstiehl, Solution algèbrique du problème de Gauss sur la permutation des points d'intersection d'une ou plusieurs courbes fermées du plan, *C.R. Acad. Sci. Paris* **283A** (1976), 551–553.
7. P. Rosenstiehl and R. C. Read, On the principal edge tripartition of a graph, *Ann. Discrete Math.* **3** (1978), 195–226.
8. J. Touchard, Sur un problème de configuration et sur les fractions continues, *Canad. J. Math.* **4** (1952), 2–25.
9. D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

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